

	$(N, w_{i \in N}, q)$ is a weighted voting game when
	es unanimity, monotonicity and the valuation
	1 when $\sum_{i \in S} w_i \ge q$ 0 otherwise
we assum	requires that $\sum_{i \in N} w_i \ge q$ . le that $\forall i \in N \ w_i \ge 0$ , monotonicity is guaranteed. of the lecture, we will assume $w_i \ge 0$ .
Ve will not $w_1, \dots, w_n$	e a weighted voting game $(N, w_{i \in N}, q)$ as ].
	voting game is a <b>succinct</b> representation, as we odefine a weight for each agent and a threshold.

dummies.

Let us consider the game [q; 4, 2, 1].

q = 1: minimal winning coalitions: {1},{2},{3}
q = 2: minimal winning coalitions: {1},{2}
q = 3: minimal winning coalitions: {1},{2,3}
q = 4: minimal winning coalitions: {1,1}
q = 5: minimal winning coalitions: {1,2},{1,3}
q = 6: minimal winning coalition: {1,2}
q = 7: minimal winning coalition: {1,2,3}
for q = 4 ("majority" weight), 1 is a dictator, 2 and 3 are

## Examples

• Let us consider the game [10; 7,4,3,3,1].

The set of minimal winning coalitions is  $\{\!\{1,2\}\!\{1,3\}\!\{1,4\}\!\{2,3,4\}\!\}$ 

Player 5, although it has some weight, is a dummy.

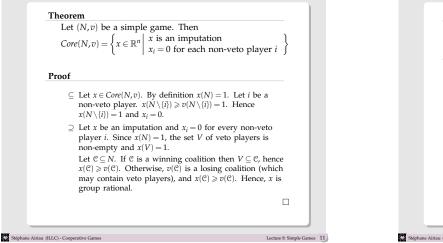
Player 2 has a higher weight than player 3 and 4, but it is clear that player 2, 3 and 4 have the same influence. • Let us consider the game [51; 49, 49, 2]

The set of winning coalition is {{1,2},{1,3},{2,3}}.

It seems that the players have symmetric roles, but it is not reflected in their weights.

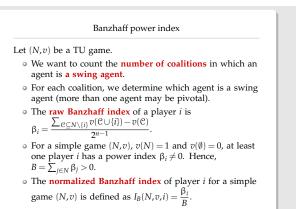
Stéphane Airiau (ILLC) - Cooperative Games

Lecture 8: Simple Games 9



Proof (continuation)  $\leftarrow \mbox{ Let } (N, v_V) \mbox{ a unanimity game. Let us prove it is a convex game. Let <math>S \subseteq N$  and  $T \subseteq N$ , and we want to prove that  $v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$ . • case  $V \subseteq S \cap T$ : Then  $V \subseteq S$  and  $V \subseteq T$ , and we have  $2 \leq 2 \checkmark$ • case  $V \nsubseteq S \cap T \land V \subseteq S \cup T$ : • if  $V \subseteq S$  then  $V \not\subseteq T$  and  $1 \leqslant 1 \checkmark$ • if  $V \subseteq T$  then  $V \not\subseteq S$  and  $1 \leqslant 1 \checkmark$ • otherwise  $V \not\subseteq S$  and  $V \not\subseteq T$ , and then  $0 \leqslant 1 \checkmark$ • case  $V \not\subseteq S \cup T$ : then  $0 \leq 0 \checkmark$ For all cases,  $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ , hence a unanimity game is convex. In addition, all members of V are veto players. П Convex simple games are the games with a single minimal winning coalition.

iphane Airiau (ILLC) - Cooperative Game 



The index corresponds to the expected marginal utility assuming all coalitions are equally likely.

Lecture 8: Simple Games 13

Weighted voting game is a strict subclass of voting games. i.e., all voting games are not weighted voting games

Example: Let  $(\{1, 2, 3, 4\}, v)$  a voting game such that the set of minimal winning coalitions is  $\{[1,2], (3,4]\}$ . Let us assume we can represent (N, v) with a weighted voting game  $[q; w_1, w_2, w_3, w_4].$ 

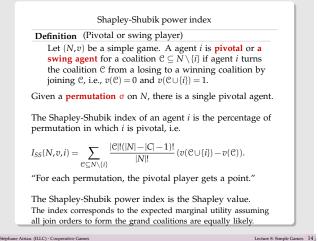
```
v(\{1,2\}) = 1 then w_1 + w_2 \ge q
v(\{3,4\}) = 1 then w_3 + w_4 \ge q
v(\{1,3\}) = 0 then w_1 + w_3 < q
v(\{2,4\}) = 0 then w_2 + w_4 < q
```

Stéphane Airiau (ILLC) - Cooperative Games

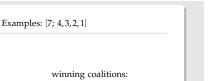
But then,  $w_1 + w_2 + w_3 + w_4 < 2q$  and  $w_1 + w_2 + w_3 + w_4 \ge 2q$ , which is impossible. Hence, (N, v) cannot be represented by a weighted voting game.✔

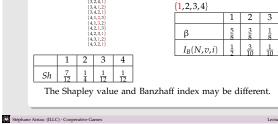
Lecture 8: Simple Games 10

## Theorem A simple game (N, v) is convex iff it is a unanimity game $(N, v_V)$ where V is the set of veto players. Proof A game is convex iff $\forall S, T \subseteq N \ v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$ . $\Rightarrow$ Let us assume (N, v) is convex. If S and T are winning coalitions, $S \cup T$ is a winning coalition by monotonicity. Then, we have $2 \le 1 + v(S \cap T)$ and it follows that $v(S \cap T) = 1$ . The intersection of two winning coalitions is a winning coalition. Moreover, from the definition of veto players, the intersection of all winning coalitions is the set V of veto players. Hence, v(V) = 1. By monotonicity, if $V \subseteq \mathbb{C}$ , $v(\mathbb{C}) = 1$ Otherwise, $V \notin \mathbb{C}$ . Then there must be a veto player $i \notin \mathbb{C}$ , and it must be the case that $v(\mathbb{C}) = 0$ Hence, for all coalition $\mathcal{C} \subseteq N$ , $v(\mathcal{C}) = 1$ iff $V \subseteq \mathcal{C}$ . П Stéphane Airiau (ILLC) - Cooperative Games Lecture 8: Simple Games 12



Stéphane Airiau (ILLC) - Co





{1,2} {1,2,3}

{**1**,**2**,**4**}

{1,3,4}

4

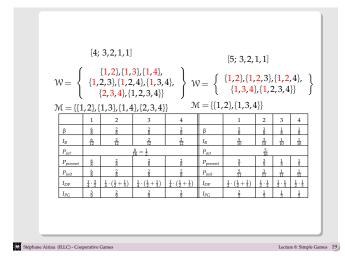
The power **to** initiate an action: 
$$P_{init}$$
 captures the power of *i* to join a losing coalition so that it becomes a winning one.

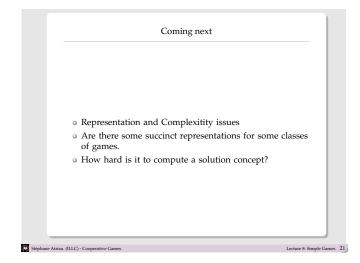
$$P_{init} = \frac{\sum_{\mathcal{C} \subseteq N \setminus \{i\}} v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})}{2^n - |W(N, v)|}.$$

Stéphane Airiau (ILLC) - Cooperative Gam

0

Lecture 8: Simple Games 17





 Maybe only minimal winning coalitions are important to measure the power of an agent (non-minimal winning coalitions may form, but only the minimal ones are important to measure power).  ${\circ}\;$  Let (N,v) be a simple game,  $i \in N$  be an agent.  $\mathcal{M}(N, v)$  denotes the set of minimal winning coalitions,  $\mathcal{M}_i(N, v)$  denotes the set of minimal winning coalitions containing i. • The **Deegan-Packel** power index of player *i* is:  $I_{DP}(N,v,i) = \frac{1}{|\mathcal{M}(N,v)|} \sum_{\mathcal{C} \in \mathcal{M}_i(N,v)} \frac{1}{|\mathcal{C}|}.$ • The **public good index** of player *i* is defined as  $I_{PG}(N,v,i) = rac{|\mathfrak{M}_i(N,v)|}{\sum_{j\in N}|\mathfrak{M}_j(N,v)|}.$ 

Lecture 8: Simple Games 18

Stéphane Airiau (ILLC) - Cooperative Game

