

	$(N, w_{i \in N}, q)$ is a weighted voting game when
	es unanimity, monotonicity and the valuation
	1 when $\sum_{i \in S} w_i \ge q$ 0 otherwise
we assum	requires that $\sum_{i \in N} w_i \ge q$. le that $\forall i \in N \ w_i \ge 0$, monotonicity is guaranteed. of the lecture, we will assume $w_i \ge 0$.
Ve will not w_1, \dots, w_n	e a weighted voting game $(N, w_{i \in N}, q)$ as].
	voting game is a succinct representation, as we odefine a weight for each agent and a threshold.

dummies.

Let us consider the game [q; 4, 2, 1].

q = 1: minimal winning coalitions: {1},{2},{3}
q = 2: minimal winning coalitions: {1},{2}
q = 3: minimal winning coalitions: {1},{2,3}
q = 4: minimal winning coalitions: {1,1}
q = 5: minimal winning coalitions: {1,2},{1,3}
q = 6: minimal winning coalition: {1,2}
q = 7: minimal winning coalition: {1,2,3}
for q = 4 ("majority" weight), 1 is a dictator, 2 and 3 are

Examples

• Let us consider the game [10; 7,4,3,3,1].

The set of minimal winning coalitions is $\{\!\{1,2\}\!\{1,3\}\!\{1,4\}\!\{2,3,4\}\!\}$

Player 5, although it has some weight, is a dummy.

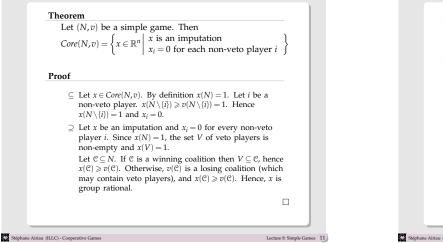
Player 2 has a higher weight than player 3 and 4, but it is clear that player 2, 3 and 4 have the same influence. • Let us consider the game [51; 49, 49, 2]

The set of winning coalition is {{1,2},{1,3},{2,3}}.

It seems that the players have symmetric roles, but it is not reflected in their weights.

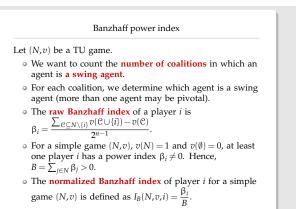
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Proof (continuation) $\leftarrow \mbox{ Let } (N, v_V) \mbox{ a unanimity game. Let us prove it is a convex game. Let <math>S \subseteq N$ and $T \subseteq N$, and we want to prove that $v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$. • case $V \subseteq S \cap T$: Then $V \subseteq S$ and $V \subseteq T$, and we have $2 \leq 2 \checkmark$ • case $V \nsubseteq S \cap T \land V \subseteq S \cup T$: • if $V \subseteq S$ then $V \not\subseteq T$ and $1 \leqslant 1 \checkmark$ • if $V \subseteq T$ then $V \not\subseteq S$ and $1 \leqslant 1 \checkmark$ • otherwise $V \not\subseteq S$ and $V \not\subseteq T$, and then $0 \leqslant 1 \checkmark$ • case $V \not\subseteq S \cup T$: then $0 \leq 0 \checkmark$ For all cases, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, hence a unanimity game is convex. In addition, all members of V are veto players. П Convex simple games are the games with a single minimal winning coalition.

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The index corresponds to the expected marginal utility assuming all coalitions are equally likely.

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Weighted voting game is a strict subclass of voting games. i.e., all voting games are not weighted voting games

Example: Let $(\{1, 2, 3, 4\}, v)$ a voting game such that the set of minimal winning coalitions is $\{[1,2], (3,4]\}$. Let us assume we can represent (N, v) with a weighted voting game $[q; w_1, w_2, w_3, w_4].$

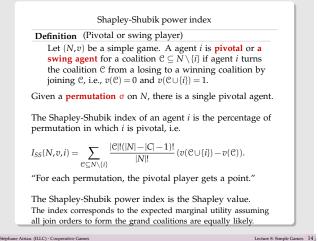
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v(\{1,2\}) = 1 then w_1 + w_2 \ge q
v(\{3,4\}) = 1 then w_3 + w_4 \ge q
v(\{1,3\}) = 0 then w_1 + w_3 < q
v(\{2,4\}) = 0 then w_2 + w_4 < q
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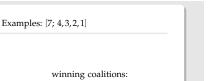
But then, $w_1 + w_2 + w_3 + w_4 < 2q$ and $w_1 + w_2 + w_3 + w_4 \ge 2q$, which is impossible. Hence, (N, v) cannot be represented by a weighted voting game.✔

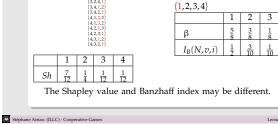
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Theorem A simple game (N, v) is convex iff it is a unanimity game (N, v_V) where V is the set of veto players. Proof A game is convex iff $\forall S, T \subseteq N \ v(S) + v(T) \leq v(S \cap T) + v(S \cup T)$. \Rightarrow Let us assume (N, v) is convex. If S and T are winning coalitions, $S \cup T$ is a winning coalition by monotonicity. Then, we have $2 \le 1 + v(S \cap T)$ and it follows that $v(S \cap T) = 1$. The intersection of two winning coalitions is a winning coalition. Moreover, from the definition of veto players, the intersection of all winning coalitions is the set V of veto players. Hence, v(V) = 1. By monotonicity, if $V \subseteq \mathbb{C}$, $v(\mathbb{C}) = 1$ Otherwise, $V \notin \mathbb{C}$. Then there must be a veto player $i \notin \mathbb{C}$, and it must be the case that $v(\mathbb{C}) = 0$ Hence, for all coalition $\mathcal{C} \subseteq N$, $v(\mathcal{C}) = 1$ iff $V \subseteq \mathcal{C}$. П Stéphane Airiau (ILLC) - Cooperative Games Lecture 8: Simple Games 12



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{1,2} {1,2,3}

{**1**,**2**,**4**}

{1,3,4}

4

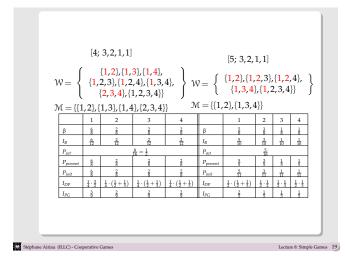
The power **to** initiate an action:
$$P_{init}$$
 captures the power of *i* to join a losing coalition so that it becomes a winning one.

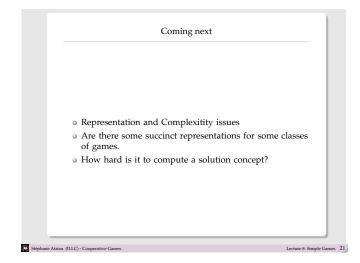
$$P_{init} = \frac{\sum_{\mathcal{C} \subseteq N \setminus \{i\}} v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})}{2^n - |W(N, v)|}.$$

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 Maybe only minimal winning coalitions are important to measure the power of an agent (non-minimal winning coalitions may form, but only the minimal ones are important to measure power). ${\circ}\;$ Let (N,v) be a simple game, $i \in N$ be an agent. $\mathcal{M}(N, v)$ denotes the set of minimal winning coalitions, $\mathcal{M}_i(N, v)$ denotes the set of minimal winning coalitions containing i. • The **Deegan-Packel** power index of player *i* is: $I_{DP}(N,v,i) = \frac{1}{|\mathcal{M}(N,v)|} \sum_{\mathcal{C} \in \mathcal{M}_i(N,v)} \frac{1}{|\mathcal{C}|}.$ • The **public good index** of player *i* is defined as $I_{PG}(N,v,i) = rac{|\mathfrak{M}_i(N,v)|}{\sum_{j\in N}|\mathfrak{M}_j(N,v)|}.$

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